Renormalization Group and Asymptotics of Solutions of Nonlinear Parabolic Equations

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Abstract

We present a general method for studying long time asymptotics of nonlinear parabolic partial differential equations. The method does not rely on a priori estimates such as the maximum principle. It applies to systems of coupled equations, to boundary conditions at infinity creating a front, and to higher (possibly fractional) differential linear terms. We present in detail the analysis for nonlinear diffusion-type equations with initial data falling off at infinity and also for data interpolating between two different stationary solutions at infinity. In an accompanying paper [5], the method is applied to systems of equations where some variables are "slaved", such as the complex Ginzburg-Landau equation.

1. Introduction

The time evolution of many physical quantities is described by nonlinear, parabolic, partial differential equations. For most of these equations, to obtain a closed form solution seems to be a hopeless task. Therefore, one tries to determine certain qualitative properties of the solution, such as its existence and regularity for all times, or its long-time asymptotics. It turns out that, for certain equations, the long-time behaviour can be predicted because the solution becomes asymptotically scale-invariant; a trivial example is given by the usual heat equation: $\dot{u} = u''$. The fundamental solution $u(x,t) =$

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$(4\pi t)^{-\frac{1}{2}}e^{-\frac{x^2}{4t}}$ is called scale invariant because its value at time t' may be obtained from its value at time t by scaling x and u. Moreover, for (suitable) initial data $u_0(x)$, the solution is asymptotically proportional to the fundamental solution, and therefore decays like $t^{-\frac{1}{2}}$. A less trivial remark is that the same asymptotic behaviour governs other equations; consider, for example, the heat equation with absorption: $\dot{u} = u'' - u^p$. Then, for $p > 3$, the long time asymptotics, given integrable initial data, is again given by the fundamental solution of the heat equation [11, 12, 16]. For long times, the nonlinear term u^p has almost no effect.

A similar situation is encountered in statistical mechanics in the theory of critical phenomena. Here too, it seems difficult to obtain an exact solution for many systems of interest. However, at the critical point, correlation functions become asymptotically scale-invariant; moreover, universality holds i.e. one may partition the set of statistical systems into different classes in such a way that, within a class, different systems have the same asymptotic behaviour at the critical point.

In the theory of critical phenomena, scale invariance and universality can both be understood on the basis of renormalization group transformations. The idea is, very roughly, as follows: start from, say, a lattice system (like the Ising model), defined by its Hamiltonian H, and fix a parameter L. Then, perform the statistical sum over all degrees of freedom corresponding to fluctuations of scale less than L. Rescale everything by L^{-1}; we obtain a new lattice system defined by a renormalised Hamiltonian $H' = R_L(H)$. This defines the renormalization group transformation. Now, iterate this procedure. The action of this transformation on the space of Hamiltonians is expected to have the following structure (this can be proven in certain cases): if we start with a Hamiltonian H which correspond to a critical point, the renormalization group will drive it towards a fixed point of that transformation. The fixed point is scale-invariant, which accounts for the asymptotic scale invariance of the original system, and a universality class corresponds to the basin of attraction of a given fixed point. Moreover, it often suffices to do a linear stability analysis of the RG transformation around the fixed point in order to determine qualitatively its basin of attraction.

From a more technical point of view, the renormalization group allows the use of perturbation theory even where it seems to fail: in spin systems, there exist high and low temperature expansions which give good results away from the critical point but fail to converge up to that point. However such perturbative methods can be used to study the RG map itself, because the latter involves only the summation over a finite number of degrees of freedom (those of scale less than L).

In 1978, Barenblatt wrote a book [1] devoted to the role of scaling (which he calls "self-similarity of the second kind") in the asymptotic behaviour of solutions of PDE's. He suggested to look at these equations using the ideas being developed in field theory and in the theory of phase transitions. However, it is only recently, that, in several very interesting papers [13, 14, 15], it was shown how the RG idea can be used to study the long-time asymptotics of parabolic PDE's. The purpose of this paper is to extend this analysis and to show that it yields new mathematical results. For example, we can show that, for a large class of perturbations of the heat equation and for small initial data, the long time behaviour of the solution is given by the fundamental solution of the heat

equation. This result, which is related to those of [11, 12, 16, 9], also holds if we replace the second derivative in the heat equation by some other derivatives.

We can also study what happens when one takes initial data that have a non trivial behaviour at infinity, i.e. such that $u_0(x)$ tends to two different stationary solutions of the PDE when x tends to infinity . We show again in a general situation, with small initial data, that the solution acquires in the long time limit a universal form, a universal "front", whose shape depends only on the boundary conditions and on the "universality class" of the equation. In an accompanying paper [5] we extend this result to the "slaving principle", i.e. to the fact that some (fast) variables in coupled systems are essentially following the behaviour of other (slow) variables. Those fast variables are an example of what is called relevant variables in RG terminology. As an example we consider in [5] the complex Ginzburg-Landau equation (see also [8, 9]).

The RG transformation for the PDE's is simply the integration of the equation up to an L-dependent time, followed by rescaling. After rescaling, we get a new equation. Upon iteration, this tends to stabilise to a fixed equation, whose scale invariant solutions are the fixed points of the RG transformation. Here also, the power of perturbation theory is enhanced by the RG approach: while the standard problem is to justify perturbation theory for an infinite time, here we use it to study the RG map, which is only a finite time problem. However, as in the theory of critical phenomena, the RG idea is not limited in principle to the perturbative regime.

We hope that the RG method will be a fruitful way to analyse a general class of equations. In any case, the equations discussed below provide very simple illustrations of rigorous applications of the RG method. We have tried to stress the analogy between the partial differential equations analysed here and technically more complicated applications of the RG like field theory (see Section 5).

2. The Renormalization Group Map

We will explain now the RG method for an equation of the type

$$\dot{u} = u'' + F(u, u', u'') \tag{1}$$

(for notational simplicity we take the spatial variable $x \in \mathbf{R}$: everything below can as well be done in $\mathbf{R}^n$). We are interested in the asymptotics of the solution of the form

$$u(x,t) \sim t^{-\frac{\alpha}{2}} f(xt^{-\frac{1}{2}}) \tag{2}$$

as $t \to \infty$.

The standard way to study this problem is, [11, 12, 16], to first look for a scale-invariant solution (if F is scale-invariant) which reduces to solving an ordinary differential equation (see Section 5) and then to establish the stability of this solution. Usually, one uses a priori estimates like the maximum principle to prove stability.

The RG method transforms the problem of large time limit into an iteration of a fixed time problem followed by a scaling transformation. The scale-invariant solution

emerges then as a fixed point of a map in the space of initial data, the RG map, and the stability analysis becomes the analysis of the stability of the fixed point under the RG. As we will see, the method does not depend on any a priori positivity properties and is applicable to systems of equations and to equations whose leading term has other derivatives.

We now explain the idea without spelling out any concrete assumptions on the F in (1) nor on the spaces and norms. The choice of the latter, as usual in non linear problems, depends on the particular problem. Thus, let us fix some (Banach) space of initial data $\mathcal{S}$. It will be convenient to take the *initial time* to be $t = 1$. Next, we pick a number $L > 1$ and set

$$u_L(x,t) = L^\alpha u(Lx, L^2 t) \tag{3}$$

where α will be chosen later and u solves (1) with the initial data $f \in \mathcal{S}$. The RG map $R : \mathcal{S} \to \mathcal{S}$ (this has to be proven!) is then

$$(Rf)(x) = u_L(x,1). \tag{4}$$

Note that u_L satisfies

$$\dot{u}_L = u_L'' + F_L(u_L, u_L', u_L''). \tag{5}$$

with $F_L(a,b,c) = L^{2+\alpha} F(L^{-\alpha}a, L^{-1-\alpha}b, L^{-2-\alpha}c)$.

We may now iterate R to study the asymptotics of (1). R depends, besides on α, on L and F. Let us denote this by $R_{L,F}$. We have then the "semigroup property"

$$R_{L^n,F} = R_{L,F_{L^{n-1}}} \circ \ldots \circ R_{L,F_L} \circ R_{L,F}. \tag{6}$$

Each R on the RHS involves a solution of a fixed time problem and the long time problem on the LHS is reduced to an iteration of these. Letting $t = L^{2n}$, we have

$$u(x,t) = t^{-\frac{\alpha}{2}} (R_{L^n,F} f)(x t^{-\frac{1}{2}}). \tag{7}$$

Now one tries to show that there exists an α such that

$$F_{L^n} \to F^* \ , \quad R_{L^n,F} f \to f^* \tag{8}$$

where

$$R_{L,F^*} f^* = f^* \tag{9}$$

is the fixed point of the RG, corresponding to the scale-invariant equation $\dot{u} = u'' + F^*$. Then, rescaling x, the asymptotics of the original problem is given by

$$u(xt^{1/2}, t) \sim t^{-\frac{\alpha}{2}} f^*(x). \tag{10}$$

We will now illustrate the RG method in two concrete cases.

3. Gaussian Fixed Point: Diffusive Repair

For $F = 0$ the equation (2.1) is just the diffusion equation and the corresponding fixed points are trivial to write down. We concentrate on integrable initial data; for a discussion of fixed points relevant to other data see the end of this section.

Going to Fourier transform and putting $\alpha = 1$, we have

$$\widehat{Rf}(k) = e^{-k^2(1-L^{-2})}\hat{f}(L^{-1}k) \equiv \widehat{R_0 f}(k). \tag{1}$$

R_0 has a line of fixed points, namely the multiples of f_0^*, with

$$\hat{f}_0^*(k) = e^{-k^2}. \tag{2}$$

f_0^* is of course the initial data of the scale invariant solution $u(x,t) = (4\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}$.

Note that R_0 has nice contractive properties, provided $\hat{f}$ has some smoothness. Indeed, let $f = f_0^* + g$ with $\hat{g}(0) = 0$. Then, $R_0 f = f_0^* + R_0 g$ and, if $\hat{g}$ is C^1, we have $|\hat{g}(\frac{k}{L})| \leq \frac{|k|}{L}|\hat{g}'(\frac{ks}{L})|$, for $0 \leq s \leq 1$, which leads to contraction in many "natural" norms. Here, we shall consider the Banach space $\mathcal{B}$ of functions f with $\hat{f} \in C^1(\mathbf{R})$, equiped with the norm (we are inspired here by [9])

$$\|f\| = \sup_k (1+k^4)(|\hat{f}(k)| + |\hat{f}'(k)|). \tag{3}$$

The $1+k^4$ factor provides smoothness for f (actually we only need $1+|k|^p$ with $p > 3$) whereas the $\hat{f}'$ term gives decay at infinity (e.g. $xf(x) \in L^2$). Now, for $\hat{g}(0) = 0$ as above, we get

$$\|R_0 g\| \leq CL^{-1}\|g\|, \tag{4}$$

with C independent of L. Here and below, C will denote a generic constant, which may change from place to place, even in the same equation. However, we shall write $C_L, C_F, \ldots$ for (generic) constants that depend on the choice of L, F etc. In the proofs, L will be chosen large enough so that e.g. $CL^{-1} < 1$. Then other quantities like ϵ in Theorem 1 below are chosen small enough in relation with L.

Next, we discuss the domain of attraction of the Gaussian fixed point (2) in the set of equations of the form (2.1). We will take the nonlinear term F to be a function $F : \mathbf{C}^3 \to \mathbf{C}$ which is analytic in a neighbourhood of the origin. Note that for a monomial $F(a,b,c) = a^{n_1}b^{n_2}c^{n_3}$ we have $F_L = L^{-d_F}F$ with

$$d_F = n_1 + 2n_2 + 3n_3 - 3. \tag{5}$$

In general, we define d_F for an analytic F by taking the smallest of the numbers (5) computed for the monomials in the Taylor series of F at 0 with non-vanishing coefficients. We call F *irrelevant*, if $d_F > 0$, *marginal*, if $d_F = 0$ and *relevant*, if $d_F < 0$.

A. The irrelevant case

We can now state the main result for an irrelevant F.

Theorem 1. *Let $F : \mathbf{C}^3 \to \mathbf{C}$ be analytic in a neighbourhood of* 0 *with* $d_F > 0$ *and fix a* $\delta > 0$. *Then there exists an* $\epsilon > 0$ *such that if* $\|f\| < \epsilon$, *the equation*

$$\dot{u} = u'' + F(u, u', u'') \quad u(x, 1) = f(x) \tag{6}$$

has a unique solution which satisfies, for some number $A = A(f, F)$,

$$\lim_{t\to\infty} t^{1-\delta} \|u(\cdot t^{\frac{1}{2}}, t) - At^{-\frac{1}{2}} f_0^*(\cdot)\| = 0. \tag{7}$$

Remark 1. As will be evident from the proof, we could as well consider a more general class of equations

$$\dot{u} = -(-\Delta)^{\beta/2} u + F \ , \quad u : \mathbf{R}^N \to \mathbf{C} \tag{8}$$

where $\widehat{(-\Delta)^{\beta/2}} f(k) \equiv |k|^\beta \hat{f}(k)$ and F is analytic in u and its partial derivatives up to order β. $\hat{f}^*(k)$ is then $e^{-|k|^\beta}$ and $u(x, t) \sim At^{-\frac{N}{\beta}} f^*(xt^{-\frac{1}{\beta}})$, provided $d_F > 0$ where, for a monomial $F = \prod_{j=1}^N \prod_{i=1}^l (\partial_j^{a_j} u)^{n_{ij}}$, we set

$$d_F = \sum_{j=1}^N \sum_{i=1}^l (N + a_i) n_{ij} - (N + \beta).$$

We restrict $a_i \leq \beta$. In (2.3) we set $\alpha = N$ and replace $L^2 t$ by $L^\beta t$. In (3), we replace k^4 by $|k|^p$ with $p > \beta + N$. In (5) above, we had $N = 1, \beta = 2$ and $a_i = 0, 1, 2$. Of course we could also take in F fractional derivatives, $|k|^\gamma$ in momentum space for $\gamma \leq \beta$.

Remark 2. The statement (7) translates in momentum space into

$$|\hat{u}(k, t) - Ae^{-tk^2}| \leq Ct^{-\frac{1}{2}+\delta}(1 + t^2 k^4)^{-1}. \tag{9}$$

Taking smoother initial data improves the fall off on the RHS accordingly. In x space, the convergence in the norm (3) implies convergence both in L^1 and in L^∞. For L^1, we use Schwartz' inequality to get

$$\|f\|_1 \leq C\|(1 + |x|)f\|_2$$

and Plancherel to bound

$$\|f\|_2 + \|xf\|_2 \leq C\|f\|$$

For L^∞, we use $\|f\|_\infty \leq \|\hat{f}\|_1 \leq C\|f\|$.

Proof. We start by discussing the local existence of the solution. Thus, turn (6) into an integral equation

$$u(t) = e^{(t-1)\partial^2} f + \int_0^{t-1} ds e^{s\partial^2} F(t - s) \equiv u_f + N(u) \tag{10}$$

where $F(\tau) = F(u(\tau), u'(\tau), u''(\tau))$ and $\partial = \frac{d}{dx}$. We solve (10) using the contraction mapping principle by introducing the Banach space of functions $u(x,t)$, $t \in [1, L^2]$ with the norm

$$\|u\|_L = \sup_{t\in[1,L^2]} \|u(t)\|. \tag{11}$$

We shall show that $T(u) = u_f + N(u)$ maps the ball $B_f = \{u | \|u - u_f\|_L \leq \|f\|\}$ into itself and is a contraction there, for $\|f\| \leq \epsilon = \epsilon(F, L)$ small.

We need to estimate $\|N(u)\|_L$. In this the use of momentum space in the norm (3) is very convenient: we are indebted to [9] for this observation. Thus, Taylor expand F in (10) and take the Fourier transform:

$$\hat{F} = \sum_{\mathbf{n}\in\mathbf{N}^3} a_{\mathbf{n}} \hat{u}^{*n_1} * \widehat{u'}^{*n_2} * \widehat{u''}^{*n_3} \tag{12}$$

where $*$ is the convolution. Inserting (12) into (10) and using the bound on the derivatives $|\widehat{u^{(l)}}(k)| \leq C\|u\|_L(1+k^2)^{-1}$, for $l = 0, 1, 2$, we get

$$\begin{aligned}\int_0^{t-1} dse^{-sk^2} |(\hat{u}^{*n_1} * \widehat{u'}^{*n_2} * \widehat{u''}^{*n_3})(k)| \leq (C\|u\|_L)^m (\frac{1 - e^{-(t-1)k^2}}{k^2})\\ \cdot \int (1 + (k - p_1)^2)^{-1} \dots (1 + (p_m)^2)^{-1} dp_1 \dots dp_m \leq\\ C_L [C\|u\|_L]^m (1 + k^4)^{-1}.\end{aligned} \tag{13}$$

with $m = n_1 + n_2 + n_3$. The same estimate holds for the derivative with respect to k and we get a convergent bound on $\|N(u)\|_L$: since F is analytic, $|a_{\mathbf{n}}| \leq (C_F)^m$, and hence

$$\|N(u)\|_L \leq C_{L,F} \|u\|_L^2 \tag{14}$$

($d_F > 0$ implies $m \geq 2$), provided $\|u\|_L$ is small enough. This holds in B_f, if $\|f\| \leq \epsilon(F, L)$, since $\|u_f\|_L \leq C\|f\|$ and so $\|u\|_L \leq (C + 1)\|f\|$. Thus T maps B_f into itself.

In the same way, we estimate

$$\|N(u_1) - N(u_2)\|_L \leq C_F L^2 (\|u_1\|_L + \|u_2\|_L) \|u_1 - u_2\|_L. \tag{15}$$

Thus, for $\|f\| \leq \epsilon(F, L)$, (10) has a unique solution in B_f

$$u(\cdot, L^2) = e^{(L^2-1)\partial^2} f + v \tag{16}$$

with

$$\|v\| \leq C_{L,F} \|f\|^2 \tag{17}$$

Now we write

$$f = A_0 f_0^* + g_0 \tag{18}$$

with $A_0 = \hat{f}(0)$. Note that $\hat{f}_0^*(0) = 1$, hence $\hat{g}_0(0) = 0$. The reason for this decomposition is that R_0 is a contraction when it acts on functions g with $\hat{g}(0) = 0$ (see (4)). Also,

$$\|g_0\| = \|f - \hat{f}(0) f_0^*\| \leq C\|f\|. \tag{19}$$

Then,

$$Rf = Lu(L\cdot, L^2) = R_0 f + Lv(L\cdot) = A_1 f_0^* + g_1 \tag{20}$$

where R_0 is defined in (1),

$$A_1 = A_0 + \hat{v}(0) \tag{21}$$

and

$$g_1 = R_0 g_0 + Lv(L\cdot) - \hat{v}(0) f_0^*, \tag{22}$$

so that $\hat{g}_1(0) = 0$. We have then the estimates, using (17),

$$|A_1 - A_0| \leq C_{L,F}\|f\|^2 \tag{23}$$

$$\|Lv(L\cdot) - \hat{v}(0) f_0^*\| \leq C_{L,F}\|f\|^2 \tag{24}$$

and, combining with (4), we get

$$\|g_1\| \leq CL^{-1}\|g_0\| + C_{L,F}\|f\|^2 \leq L^{-(1-\delta)}\|f\| \tag{25}$$

for $\|f\| \leq \epsilon(L, F)$ and L large (depending on δ).

The proof is now completed by iterating this procedure. Set

$$f_n \equiv R_{L^n,F} f = A_n f_0^* + g_n \tag{26}$$

So, with v above equal v_0,

$$A_{n+1} = A_n + \hat{v}_n(0) \tag{27}$$

$$g_{n+1} = R_0 g_n + Lv_n(L\cdot) - \hat{v}_n(0) f_0^* \tag{28}$$

and assume inductively that

$$|A_n| \leq (C - L^{-n})\|f\| \quad , \quad \|g_n\| \leq CL^{-(1-\delta)n}\|f\|, \tag{29}$$

so that $\|f_n\| \leq C\|f\|$. Then, repeating the above analysis and noting that in (14) and (15), C_F is replaced by $L^{-nd_F} C_F$ (see (2.5)), we get, instead of (17), (23), (25),

$$\|v_n\| \leq C_{L,F} L^{-nd_F}\|f\|^2 \tag{30}$$

$$|A_{n+1} - A_n| \leq C_{L,F} L^{-nd_F}\|f\|^2 \tag{31}$$

$$\|g_{n+1}\| \leq CL^{-1}\|g_n\| + C_{L,F} L^{-nd_F}\|f\|^2. \tag{32}$$

Thus, (29) iterates ($d_F \geq 1$) and we get that $A_n \to A$ with $|A_n - A| \leq C_{L,F} L^{-nd_F} \|f\|^2 \leq CL^{-nd_F}\|f\|$ for $\|f\|$ small,

$$|A - \hat{f}(0)| = |A - A_0| \leq C_{L,F}\|f\|^2 \tag{33}$$

and (see (2.7), (26)), for $t = L^{2n}$,

$$\|u(\cdot t^{1/2}, t) - At^{-\frac{1}{2}} f_0^*(\cdot)\| \leq Ct^{-1+\frac{1}{2}\delta}\|f\|. \tag{34}$$

It is trivial to extend this bound to $t = \tau L^{2n}$, with $1 \leq \tau \leq L^2$ by replacing everywhere L^2 by τL^2. This proves the claim. □

B. The marginal case

Let us now consider the marginal cases, i.e.

$$F = -u^3 + G(u, u', u'') \tag{35}$$

or

$$F = 2uu' + H(u, u', u'') = (u^2)' + H(u, u', u'') \tag{36}$$

where G and H are irrelevant. (36) is just the Burgers equation with a perturbation. Since the discussion of that equation uses some ideas introduced in the next section, we shall discuss it at the end of Section 4. In (35), we need the negative sign: for $G = 0$ and a positive sign, any non zero initial data leads to a solution that blows up in a finite time [18, 9]. However, we do not have to assume that the initial value is pointwise positive; rather we want f to be near a small multiple of f^*. Although a general analytic irrelevant G could be treated, we assume for simplicity that the Taylor expansion of G starts at degree 4 or higher. Without this assumption, in the proof below, we would need to consider first a "crossover" time during which the G term would dominate u^3. We will therefore scale u by $\lambda^{\frac{1}{2}}$, where λ is chosen so that $\hat{u}(\cdot, 1)(0) = 1$, and consider the equation

$$\dot{u} = u'' - \lambda u^3 + G_\lambda(u, u', u'') \tag{37}$$

with $G_\lambda(z) = \lambda^{-\frac{1}{2}} G(\lambda^{\frac{1}{2}} z)$ (which is of order $\lambda^{3/2}$ for λ small) and initial data

$$f = f_0^* + g\ ,\ \ \hat{g}(0) = 0. \tag{38}$$

We have then

Theorem 2. *For any $\delta > 0$ there exist λ_0 , $\epsilon > 0$ such that for $0 < \lambda \leq \lambda_0$ and $\|g\| \leq \epsilon$, we have*

$$\lim_{t\to\infty} t^{\frac{1}{2}} (\log t)^{1-\delta} \|u(\cdot t^{\frac{1}{2}}, t) - (\frac{\lambda}{2\sqrt{3}\pi} t \log t)^{-\frac{1}{2}} f_0^*(\cdot)\| = 0 \tag{39}$$

Proof. We proceed as in the proof of Theorem 1. We use (10) and we get, for λ small, instead of (14),

$$\|N(u)\|_L \le C_{L,G}\lambda \tag{40}$$

($\|u\|_L$ is of order one here). This, with the analogue of (15), proves local existence, leading to

$$Rf = R_0 f + Lv(L\cdot)$$

with

$$\|v\| \le C_{L,G}\lambda \tag{41}$$

Since u^3 is marginal, we have to handle rather explicitely its main effect on u which will lead to the logarithmic correction in (39). Let, in (10), $N(u) = -\lambda N_3(u) + N_G(u)$ corresponding to the two terms in (37). So,

$$N_3(u)(t) = \int_0^{t-1} ds e^{s\partial^2}(u(t-s))^3$$

We have

$$\|N_G(u)\|_L \le C_{L,G}\lambda^{3/2} \tag{42}$$

Denote by u_A^* the solution of (37) with $G_\lambda = 0$ and $f = Af_0^*$, and by u_A the solution of (37) with $f = Af_0^* + g$. So,

$$u_A^*(t) = Ae^{(t-1)\partial^2} f_0^* - \lambda N_3(u_A^*)(t) \tag{43}$$

$$u_A(t) = Ae^{(t-1)\partial^2} f_0^* + e^{(t-1)\partial^2} g - \lambda N_3(u_A)(t) + N_G(u_A)(t) \tag{44}$$

Here $A = 1$, but we shall use below the following bound, for $|A| \le 1$ and $\|g\| \le \epsilon$:

$$\|N_3(u_A) - N_3(u_A^*)\|_L \le C_L(A^2\|g\| + \|g\|^3) + C_{L,G}\lambda^{\frac{3}{2}}. \tag{45}$$

To prove (45), first show, like in (15), but with a cubic nonlinearity, that

$$\|N_3(u_A) - N_3(u_A^*)\|_L \le C_L(\|u_A\|_L + \|u_A^*\|_L)^2\|u_A - u_A^*\|_L + C_{L,G}\lambda^{\frac{3}{2}}$$

Then, use (42), (43), (44) to show

$$\|u_A - u_A^*\|_L \le C_L\|g\| + C_{L,G}\lambda^{\frac{3}{2}} + \lambda\|N_3(u_A) - N_3(u_A^*)\|_L$$

Finally, use $\|u_A^*\|_L \le C_L|A|$ and $\|u_A\|_L \le C_L(|A| + \|g\|)$, which follows from (43), (44); (45) follows for λ small.

If we define

$$v^* = \int_0^{L^2-1} ds e^{s\partial^2}(e^{(L^2-s-1)\partial^2} f_0^*)^3$$

we get, inserting (43) in $N_3(u_A^*)$, and using the bound $\|N_3(u_A^*)\|_L \leq C_L|A|^3$,

$$\|N_3(u_A^*)(L^2) - A^3 v^*\| \leq C_L \lambda A^5 \tag{46}$$

for $|A| \leq 1$. Now, write

$$v = -\lambda v^* + w \tag{47}$$

where, using (42), (45),(46), for $A = 1$ and $\|g\| \leq \epsilon$,

$$\|w\| \leq C_{L,G}\lambda(\|g\| + \lambda^{\frac{1}{2}}). \tag{48}$$

((46) gives a term of order λ^2 here). Again, we write

$$f_1 \equiv Rf = A_1 f_0^* + g_1 \tag{49}$$

with $A_1 = 1 + \hat{v}(0)$ ($A_0 = 1$ here, by (38)) and g_1 given by (22). Denote $\hat{v}^*(0) = \beta$. We get from (48)

$$|A_1 - 1 + \lambda\beta| = |\hat{w}(0)| \;\; \leq \;\; C_{L,G}\lambda(\|g\| + \lambda^{\frac{1}{2}}) \tag{50}$$

and, from (4), (41),

$$\|g_1\| \;\; \leq \;\; CL^{-1}\|g\| + C_{L,G}\lambda \tag{51}$$

The iteration is as follows: the G term will run down with a factor L^{-nd_G}, which improves the bound (42) and the last term in (45), but we shall see that, unlike the situation of Theorem 1, A_n will go to zero and we have to keep track of the correction of order A^5 in (46). We use (26), (27), (28) and we have

$$v_n = -\lambda A_n^3 v^* + w_n \tag{52}$$

with, using again (42), (45), (46),

$$\|w_n\| \leq C_{L,G}\lambda(A_n^2\|g_n\| + \|g_n\|^3 + \lambda^{\frac{1}{2}} L^{-nd_G} + \lambda A_n^5). \tag{53}$$

So,

$$|A_{n+1} - A_n + \lambda\beta A_n^3| \;\; \leq \;\; C_{L,G}\lambda(A_n^2\|g_n\| + \|g_n\|^3 + \lambda^{\frac{1}{2}} L^{-nd_G} + \lambda A_n^5) \tag{54}$$

Using the fact that $\beta > 0$ (see (58) below), we see that A_n decreases and so, using (28), (4), (52), (53), $\|v^*\| \leq C$, and $C_{L,G}\lambda A_n^2 \leq L^{-1}$, for λ small, we get

$$\|g_{n+1}\| \;\; \leq \;\; CL^{-1}\|g_n\| + C_{L,G}\lambda(A_n^3 + \|g_n\|^3 + \lambda^{\frac{1}{2}} L^{-nd_G} + \lambda A_n^5). \tag{55}$$

The iteration of A_n, g_n, leads to

$$A_n^2 = (2\lambda\beta n + b_n)^{-1} \tag{56}$$

$$\|g_n\| \ \le \ C_{L,G} n^{-\frac{3}{2}} \tag{57}$$

for $n \geq 1$, with $|b_{n+1} - b_n| \leq C_{L,G} n^{-\frac{1}{2}}$, which gives $|b_n| \leq C_{L,G}\sqrt{n}$. Indeed, the leading term in (55) is A_n^3 and, in (54), $A_n^2\|g_n\| + A_n^5$. We get $A_n = (2\lambda\beta n)^{-\frac{1}{2}} + \mathcal{O}(n^{-1})$. Finally, we compute

$$\beta = \frac{1}{4\pi^2}\int_0^{L^2-1} ds \int_{\mathbf{R}^2} dpdq e^{(s-L^2)(p^2+q^2+(p-q)^2)} = \frac{\log L}{2\sqrt{3}\pi}. \tag{58}$$

Using $2n \log L = \log t$, the claim follows as in Theorem 1. $\square$

4. Non-Gaussian fixed point: stability of a front

If equation (2.1) has other (constant) stationary solutions than $u = 0$, we may study the problem where $\lim_{x\to\pm\infty} u(x, 1)$ takes two different values. This corresponds to a *front* in the initial data. One may then inquire about the stability of this front and the universal features of the long time solution in such a situation. These problems are discussed in general in [2, 7] and, for coupled equations related to the equations discussed here, see [9, 8, 5]. Here, the restriction to one space dimension is essential.

An example of such an equation is the nonlinear diffusion equation

$$\dot{u} = \partial((1 + a(u, \partial u))\partial u) \ \ , \ \ u(x, 1) = \phi(x) \ \ , \ \ \lim_{x\to\pm\infty} \phi(x) = u_\pm \tag{1}$$

where we denote $\partial = \frac{d}{dx}$. This is a special case of (2.1), with $F = au'' + \frac{\partial a}{\partial u}(u')^2 + \frac{\partial a}{\partial u'}u'u''$. If a is analytic and if $a(0, 0) = 0$, F is irrelevant. (1) has $u = $ const. as a stationary solution.

Remark 1. The most general such equation in one variable we could deal with is

$$\dot{u} = (1 + a(u, u', u''))u'' + b(u, u', u'')(u')^2 \tag{2}$$

with a and b analytic as before. See Remark 2 below.

To understand the problem in the RG setup, let us first consider the trivial case $a = 0$. This is of course exactly soluble. We have

$$\begin{aligned} u(\sqrt{t}x, t+1) &= \frac{1}{\sqrt{4\pi}}\int e^{-\frac{1}{4}(y-x)^2}\phi(\sqrt{t}y)dy \\ &\to_{t\to\infty} u_- + (u_+ - u_-)e(x) \equiv \phi_0^*(x) \end{aligned} \tag{3}$$

where $e(x) = \int_{-\infty}^{x} e^{-\frac{1}{4}y^2} \frac{dy}{\sqrt{4\pi}}$. In RG terminology, we have the "Gaussian" fixed point ϕ_0^* corresponding to the $u_\pm$ boundary condition problem. It is easy to check that ϕ_0^* is a fixed point for the map

$$R_L \phi = u(L\cdot, L^2) \quad , \quad u(\cdot, 1) = \phi \tag{4}$$

Note the absence of the multiplicative L factor in R_L (in (2.3), we have $\alpha = 0$): the initial data are not normalizable by their integral and $u(x,t)$ does not, in general, go to zero as t goes to infinity.

The stability of this fixed point is also easy to understand. We write $\phi = \phi_0^* + f$, where $f(\pm\infty) = 0$. Then $\dot{v} = v''$, with $v(x,1) = f(x)$, and the analysis of the previous section applies. Thus we expect

$$u(x,t) \sim \phi_0^*(\frac{x}{\sqrt{t}}) + \frac{\hat{f}(0)}{\sqrt{t}} f_0^*(\frac{x}{\sqrt{t}}) + \mathcal{O}(t^{\delta-1}) \tag{5}$$

This is the asymptotics we wish to prove for (1), except that ϕ_0^* and f_0^* are replaced by non trivial fixed points. We will prove a theorem for small data, so $u_\pm$ will be taken small. What we do below is an (elementary) example of what is called, in the theory of critical phenomena, the "epsilon expansion". Before stating the precise results, let us do the heuristics.

Upon scaling $u_L(x,t) = u(Lx, L^2 t)$, u_L satisfies (1) with a replaced by $a_L(u, u') = a(u, L^{-1}u')$. Thus, we will search for a fixed point for the RG (4) with u satisfying the equation obtained in the limit $L \to \infty$:

$$\dot{u} = \partial((1 + a^*(u))\partial u) \quad , \quad u(x,1) = \phi^*(x) \tag{6}$$

with $\phi^*(\pm\infty) = u_\pm$ and $a^*(u) = a(u, 0)$. We will find the fixed point ϕ^* by finding a scale invariant solution to (6):

$$u(x,t) = \phi^*(\frac{x}{\sqrt{t}}). \tag{7}$$

We get (replacing $\frac{x}{\sqrt{t}}$ by x)

$$\partial((1 + a^*(\phi^*))\partial\phi^*) + \frac{1}{2}x\partial\phi^* = 0 \tag{8}$$

and we look for a solution

$$\phi^* = \phi_0^* + \psi \tag{9}$$

with $\psi(\pm\infty) = 0$ and ϕ_0^* is the Gaussian solution (3). Thus, we get for ψ the equation

$$A\psi = \partial(a^*(\phi_0^* + \psi)\partial(\phi_0^* + \psi)) \tag{10}$$

with

$$A = -\partial^2 - \frac{1}{2}x\partial. \tag{11}$$

Let us also denote

$$\psi_0 = \partial(a^*(\phi_0^*)\partial\phi_0^*). \tag{12}$$

We shall discuss the properties of A^{-1} later. We solve the fixed point equation (10) in the space of C^N functions equipped with the norm

$$\|\psi\|_N = \max_{0\le m\le N} \sup_x |\partial^m\psi(x)|e^{\frac{x^2}{8}}. \tag{13}$$

Let the degree of a^* (i.e. $a^*(u) = \mathcal{O}(u^d)$ for u small) be $d > 0$. Then we have the

Proposition. *Let $a : \mathbf{C}^2 \to \mathbf{C}$ in (1) be analytic in a neighbourhood of 0 and let N be an integer. Then there exists an $\epsilon > 0$ such that, for $|u_\pm| \le \epsilon$ in (3), (10) has a unique solution with*

$$\|\psi - \psi_0\|_N \le C\epsilon^{2d+1}.$$

The proof of the proposition will be given after the one of Theorem 3 below. We will also see that ψ_0 can be written down explicitely and is $\mathcal{O}(\epsilon^{d+1})$ in the norm (13).

Given ϕ^*, we return to equation (1), and write

$$u(x,t) = \phi^*(\frac{x}{\sqrt{t}}) + v(x,t) \equiv u^*(x,t) + v(x,t). \tag{14}$$

Because of (6, 7), v satisfies the equation

$$\dot{v} = \partial((1+a)\partial v + (a - a^*(u^*))\partial u^*) \ \ , \ \ v(x,1) = f(x) \tag{15}$$

with $\lim_{x\to\pm\infty} f(x) = 0$. Now the RG is as in Section 3:

$$R_L f = Lv(L\cdot, L^2) \equiv v_L(\cdot, 1) \tag{16}$$

However, the fixed point will not be the Gaussian f_0^* of that Section, but a different one. We find it by taking the scaling limit: v_L satisfies the equation

$$\dot{v_L} = \partial((1+a_L)\partial v_L + L(a_L - a^*(u^*))\partial u^*) \tag{17}$$

with $a_L = a(u^* + L^{-1}v_L, L^{-1}\partial(u^* + L^{-1}v_L))$, and therefore we look for a fixed point satisfying the linear equation obtained from (17) by letting $L \to \infty$:

$$\dot{v}^* = \partial(\partial((1 + a^*(u^*))v^*) + \frac{\partial a}{\partial u'}(u^*, 0)(\partial u^*)^2). \tag{18}$$

This is solved by setting

$$v^*(x,t) = \frac{1}{\sqrt{t}} f^*(\frac{x}{\sqrt{t}}) \tag{19}$$

whence, $\partial(\partial((1+a^*(\phi^*))f^*)+\frac{1}{2}xf^*+h)=0$ where $h=\frac{\partial a}{\partial u'}(\phi^*,0)(\partial\phi^*)^2$. We solve this equation by integrating once (we shall look for an integrable f^*, so the constant of integration is zero) and then solving a first order equation:

$$f^*(x)=(\int_0^x h(y)F(y)dy+\mathcal{N})e^{-\frac{1}{2}\int_0^x\frac{y+2\partial a^*(\phi^*(y))}{1+a^*(\phi^*(y))}dy}. \tag{20}$$

where $F(x)=-(1+a^*(\phi^*(x))^{-1}e^{+\frac{1}{2}\int_0^x\frac{y+2\partial a^*(\phi^*(y))}{1+a^*(\phi^*(y))}dy}$. For future convenience, we normalize this by choosing $\mathcal{N}$ such that $\int f^*=\hat{f}^*(0)=1$. Note that f^* is, for ϕ^* small, a small perturbation of f_0^*. In particular, f^* is smooth and decays rapidly at infinity.

Remark 2. This is the only place where (1) is simpler than (2). For (2), (18) is replaced by

$$\dot{v}^*=(1+a^*(u^*))\partial^2v^*+2b^*(u^*)\partial u^*\partial v^*+(\frac{\partial a^*}{\partial u^*}(u^*)(\partial^2u^*)+$$
$$\frac{\partial b^*}{\partial u^*}(u^*)(\partial u^*)^2)v^*+(\frac{\partial a}{\partial u'}(u^*,0,0)(\partial^2u^*)+\frac{\partial b}{\partial u'}(u^*,0,0)(\partial u^*)^2)(\partial u^*) \tag{21}$$

with $a^*(u)=a(u,0,0)$ and b^* similarily. v^* will not be as explicit as (19,20) and $\frac{1}{\sqrt{t}}$ in front of f^* in (19) will be replaced by $t^{(-\frac{1}{2}+\mathcal{O}(\epsilon))}$ since, while $\int v^*$ is constant for the solution of (18) (integrate both sides), it is not conserved by (21); see [13, 14] for a discussion of a similar effect.

Before stating the main result of this section, we need to specify the space of initial data ϕ of (1), i.e. the f of (15). We can not directly use the norms (3.3) introduced in Section 3. The reason is that the function ϕ^* in (9) involves the Gaussian fixed point (3) which has no falloff at infinity. In particular it is not in the Banach space we used before. However, the only way u^* enters in (15) is as ∂u^* that falls off like a gaussian, or by multiplying functions that we expect to fall off at infinity. As a consequence of the latter possibility, we may not use pure momentum space bounds as in (3.13): the convolutions would then involve $\widehat{u^*}$ which is too singular ($|\hat{\phi}_0^*|\simeq|k|^{-1}$ for small k). Instead, we introduce a norm that encodes more sharply than (3.3) both the long and short distance properties of the solution.

Thus, let χ be a non negative C^∞ function on $\mathbf{R}$ with compact support on the interval $(-1,1)$, such that its translates by $\mathbf{Z}$, $\chi_n=\chi(\cdot-n)$, form a partition of unity on $\mathbf{R}$. For $f\in C^2$, we then introduce the norm

$$\|f\|=\sup_{n\in\mathbf{Z},k\in\mathbf{R},i\le 2}(1+n^4)(1+k^2)|\widehat{\chi_n\partial^i f}(k)|. \tag{22}$$

Roughly, $\|f\|<\infty$ means that f falls off at least as x^{-4} at infinity and $\hat{f}(k)$ as k^{-2}. Note that the derivatives of ϕ^* and f^* and its derivatives have a finite norm. Comparing with (3.3), we have k^2 instead of k^4 but two derivatives act on f. We used k^2 to do the convolutions in (3.13). The n^4 could be changed to anything increasing not faster than $e^{\frac{n^2}{8}}$ (coming from (13)). These norms for different choices of χ may be shown to be equivalent.

Theorem 3. *Let $a : \mathbf{C}^2 \to \mathbf{C}$ be analytic in a neighbourhood of* 0 *with positive degree and fix a* $\delta > 0$*. There is an* $\epsilon > 0$ *such that for* $|u_\pm|$, $\|f\| \leq \epsilon$*, the equation*

$$\dot{u} = \partial(1 + a(u, \partial u)\partial u) \ \ , \ \ u(x, 1) = \phi^*(x) + f(x) \tag{23}$$

has a unique solution, satisfying

$$\lim_{t\to\infty} t^{1-\delta} \|u(\sqrt{t}\cdot, t) - \phi^*(\cdot) - \frac{\hat{f}(0)}{\sqrt{t}} f^*(\cdot)\| = 0 \tag{24}$$

where f^* *is given in* (20) *and* ϕ^* *in the Proposition.*

Remark 3. The convergence in the norm (22) again implies convergence in L^1 and in L^∞, see equations (39) and (40) below, applied to $i = 0$. Equation (24) means that $u(x, t)$ behaves, for $|x| \leq C$ and $t \to \infty$ as $\phi^*(0) + At^{-\frac{1}{2}} + \mathcal{O}(t^{\delta-1})$, where $A = (\phi^*)'(0) + \hat{f}(0)f^*(0)$. Thus, locally, the solution goes to a constant i.e. a stationary solution of (1), with a diffusive correction. This solution is "selected" by the boundary condition at infinity, $u_\pm$, and by a.
Remark 4. Since ψ in (9) satisfies $\|\psi\| \leq C\epsilon^2$ (this follows easily from the Proposition) we can replace ϕ^* in (23) by ϕ_0^*, given by (3), which makes the hypothesis more explicit.

Proof. We consider the equation (15) and the RG (16). It is convenient to separate the f^* piece from f. We write

$$f = \hat{f}(0)f^* + g \tag{25}$$

and correspondingly

$$v = v^* + w \tag{26}$$

with

$$v^*(x, t) = \frac{\hat{f}(0)}{\sqrt{t}} f^*(\frac{x}{\sqrt{t}}). \tag{27}$$

First, we have $|\hat{f}(0)| \leq \sum_{n\in\mathbf{Z}} |\widehat{\chi_n f}(0)| \leq \sum_{n\in\mathbf{Z}} \frac{\|f\|}{1+n^4} \leq C\|f\|$ and $\|f^*\| \leq C$, which implies

$$\|g\| \leq C\|f\| \leq C\epsilon. \tag{28}$$

The equation we finally study is the one satisfied by w:

$$\dot{w} = \partial^2 w + K \ \ , \ \ w(x, 1) = g(x) \tag{29}$$

where $\hat{g}(0) = 0$ (see (25) and the normalisation in (20)) and where (see (15), (18))

$$K = \partial[(a - a^*(u^*))\partial v^* + a\partial w + (a - a^*(u^*) - \frac{\partial a}{\partial u'}(u^*, 0)(\partial u^*))\partial u^* - v^*\partial a^*(u^*)] \tag{30}$$

with $a = a(u, u')$, $u = u^* + v^* + w$. The explicit form of K is not important for us, all we need to know is that it is given as a convergent power series in u^*, v^*, w and their derivatives up to second order, has no linear term and no term which is only a power of u^*. We shall also use some scaling properties of K discussed below.

In RG language, we need to control (see(16))

$$L^n w(L^n x, L^{2n}) = R_{L^n,K} g = R_{L,K_{n-1}} \dots R_{L,K} g \tag{31}$$

where now K_{n+1} is obtained from K_n by replacing (see (16), (4))

$$\partial^i v^* \to L^{-1-i} \partial^i v^* \ , \ \partial^i w \to L^{-1-i} \partial^i w \ , \partial^i u^* \to L^{-i} \partial^i u^*. \tag{32}$$

To see the behaviour of the terms in (30) under this scaling, define $d = n_\partial + n_{v^*} + n_w - 3$ where n_{v^*}, n_w count the number of v^*, w factors and their derivatives, while n_∂ is the total number of derivatives. We say that a term in (30) is irrelevant if $d > 0$ and marginal if $d = 0$. A term with $d < 0$ would be relevant, but these terms have been included in the equations for u^* and v^*. To check this, notice that each term in (30) has either more than two derivatives or two derivatives and at least one v^* or w factor, so that $d \geq 0$. In fact, the only marginal terms in (30) are of the form $(u^*)^l \partial^2 w$ or $(u^*)^l \partial u^* \partial w$. To check that there are no terms like that with w replaced by v, observe that such terms are cancelled because of the $a - a^*$ in front of ∂v^* and because of the subtraction between the two last terms in (30). A term like $\partial \frac{\partial a}{\partial u'}(u^*, 0)(\partial u^*)^2$ is also marginal, but it is cancelled by the subtraction in the factor multiplying ∂u^* in (30).

Eventually, using $\hat{g}(0) = 0$, we will show that $\|R_{L^n,K} g\|$ goes to zero like $L^{-(1-\delta)n}$. Using (14), (26), (27), (31), this will prove (24). We split the proof again into three steps: the proof of local existence, the control of R and the iteration of R.

(a) *Local existence.* We have to solve the integral equation

$$w(t) = e^{(t-1)\partial^2} g + \int_0^{t-1} ds e^{s\partial^2} K(t - s)$$

(with obvious notation) using the contraction mapping principle with the norm

$$\|w\|_L = \sup_{t \in [1, L^2]} \|w(\cdot, t)\|. \tag{33}$$

As in Theorem 1, we expand K as a power series and estimate a generic term

$$\alpha(x) = \int_0^{t-1} ds \int dy e^{s\partial^2}(x - y) u^*(y)^l F(y)$$

where $F(y)$ is a (non empty) product of the functions v^*, w and their derivatives, and derivatives of u^*, and $l \geq 0$. We supressed the $t - s$ variable in u^* and F. We localize the y variable:

$$\alpha(x) = \sum_{m \in \mathbf{Z}} \int_0^{t-1} ds \int dy e^{s\partial^2}(x - y) \chi_m(y) u^*(y)^l F(y) \equiv \sum_{m \in \mathbf{Z}} \alpha_m(x). \tag{34}$$

We want to bound

$$\beta_{mn} = \sup_k |(1+k^2)\chi_n \widehat{\partial^i \alpha_m}(k)|. \tag{35}$$

We distinguish between $|n-m| \geq 2$ and $|n-m| < 2$.

(A) Let first $|m-n| \geq 2$. Then χ_m and χ_n have disjoint supports and $e^{s\partial^2}(x-y)$ is smooth uniformly in s. We write

$$\beta_{mn} = \sup_k |\int dx e^{-ikx} \int_0^{t-1} ds \int dy (1-\partial_x^2)(\chi_n(x)\partial_x^i e^{s\partial^2}(x-y))\chi_m(y) u^*(y)^l F(y)| \tag{36}$$

and estimate the various factors on the RHS.

First, we use, for $j \leq 4$,

$$|\partial^j e^{s\partial^2}(x-y)| \leq C e^{-|m-n|} \tag{37}$$

on the support of χ_m and χ_n. To bound $F(y)$, we need the bounds (for $0 \leq i \leq 2$)

$$\|\partial^i u^*\|_\infty \,, \; \|\partial^i v^*\|_\infty \leq C\epsilon \tag{38}$$

which follow from (3), the Proposition and (20,27,28), and the bound

$$\begin{aligned} |\partial^i w(x)| &= \sum_n |\chi_n(x)\partial^i w(x)| \leq \sum_n \int dk |\widehat{\chi_n \partial^i} w(k)| \\ &\leq \sum_n (1+n^4)^{-1} \int dk (1+k^2)^{-1} \|w\| \leq C\|w\| \end{aligned} \tag{39}$$

To extract the factor $(1+m^4)^{-1}$, we show that

$$\int |\chi_m(y)\partial^i w(y)| dy \leq \frac{C\|w\|}{1+m^4} \tag{40}$$

and similarily for v^* and u^* (with $i > 0$ for the latter) with $\|w\|$ replaced by ϵ.

Indeed, let $\phi_m \in C_0^\infty(\mathbf{R})$ be such that $\phi_m \chi_m = \chi_m$. Then

$$\begin{aligned} \int |\chi_m \partial^i w| &= \int |\chi_m \phi_m \partial^i w| \leq \left(\int \phi_m^2\right)^{\frac{1}{2}} \\ \cdot \left(\int |\chi_m \partial^i w|^2\right)^{\frac{1}{2}} &\leq C\left(\int |\widehat{\chi_m \partial^i} w|^2\right)^{\frac{1}{2}} \leq \frac{C\|w\|}{1+m^4} \end{aligned} \tag{41}$$

For v^* and u^* (with $i > 0$ for the latter) the bound holds with $\|w\|$ replaced by ϵ due to the explicit expressions and bounds (3), (20) and the Proposition.

Now we bound (36): the x integral is controlled by $\chi_n(x)$ or its derivatives, the s integral is less than L^2 and we use (38) and (39) for $u^*(y)^l$ and all factors in $F(y)$ except one, for which we use (40):

$$\beta_{nm} \leq L^2 C^{l+M+N} e^{-|m-n|}(1+m^4)^{-1} \epsilon^{l+M} \|w\|_L^N \tag{42}$$

with M the total number of factors of v^* and its derivatives and of derivatives of u^* in F, and N similarily for w. We have $l + M + N \geq 2$.

(B) Let now $|m - n| < 2$. The difficulty is that we do not have (37) for s close to zero. But we do not have to control a sum over $m \in \mathbf{Z}$, and we can use Fourier transforms. Let us denote $\phi_m u^{*l}$ by f_m where ϕ_m is as in (A). Then

$$\chi_n \widehat{\partial^i \alpha_m}(k) = \int dsdpdq \hat{\chi}_n(k-p)(ip)^i e^{-sp^2} \hat{f}_m(p-q) \widehat{\chi_m F}(q). \tag{43}$$

Let us consider the various factors on the RHS. Since χ is C^∞ with compact support, we have

$$|\hat{\chi}_n(k-p)| = |e^{-i(k-p)n} \hat{\chi}(k-p)| \leq C_l(1 + |k-p|^l)^{-1} \tag{44}$$

for any l.

For $\hat{f}_m$, note that

$$\int |(1 + (-\partial^2)^r)\phi_m u^{*l}(x)|dx \leq C_r(C\epsilon)^l \tag{45}$$

for all r, whence

$$|\hat{f}_m(p-q)| \leq C_r(C\epsilon)^l(1 + (p-q)^{2r})^{-1}. \tag{46}$$

Also, $\int ds|p|^j e^{-sp^2} \leq CL^2$ if $j \leq 2$, so, provided we can show

$$|\widehat{\chi_m F}(q)| \leq C^{N+M}\epsilon^M \|w\|^N (1 + m^4)^{-1}(1 + q^2)^{-1} \tag{47}$$

we can perform the convolutions in (43) to get

$$\beta_{mn} \leq L^2 C^{l+N+M}(1 + m^4)^{-1}\epsilon^{l+M}\|w\|_L^N. \tag{48}$$

Using (42),(48), and the contraction mapping principle as in the proof of Theorem 1, we find a solution $w(x,t)$ such that

$$\|w(\cdot, L^2) - e^{(L^2-1)\partial^2}g(\cdot)\| \leq C_{L,a}\epsilon(\epsilon + \|g\|). \tag{49}$$

The first term comes from $N = 0, l + M \geq 2$ and the second from $l + M \geq 1, N \geq 1$. To apply the contraction mapping principle, we used $\|e^{(t-1)\partial^2}g\|_L \leq C\|g\|$ which is proven like (52) below.

To prove (47), use

$$|\widehat{\chi_m \partial^i w}| \leq (1 + m^4)^{-1}(1 + k^2)^{-1}\|w\|$$

and

$$|\widehat{\partial^i w}(k)| = |\sum_n \chi_n \widehat{\partial^i w}(k)| \leq \frac{C\|w\|}{1 + k^2}$$

together with similar bounds for v^* and the derivatives of u^* and perform the convolutions as in equation (3.13).

(b) *Contraction.* Let us now turn to the RG (29), i.e.

$$Rg = Lw(L\cdot, L^2). \tag{50}$$

Note that because of the derivative in K (see (30)), R preserves the condition $\hat{g}(0) = 0$ i.e., by (29), $\frac{d}{dt}\int w(x,t)dx = 0$ and so, $\int (Rg)(x)dx = \int g(x)dx = 0$. We will show that

$$\|Rg\| \leq \frac{C}{L}\|g\| + C_{L,a}\epsilon(\epsilon + \|g\|). \tag{51}$$

Indeed, by (49), all we need is to show

$$\|R_0 g\| = \|L(e^{(L^2-1)\partial^2}g)(L\cdot)\| \leq \frac{C}{L}\|g\|. \tag{52}$$

It is now easier to work in the x representation. We have

$$R_0 g = \int G(x,y)g(y)dy \tag{53}$$

with

$$G(x,y) = (4\pi(1-L^{-2}))^{-1}e^{-\frac{1}{4}(1-L^{-2})^{-1}(x-L^{-1}y)^2}. \tag{54}$$

We need to bound

$$\|R_0 g\| = \sup_{k,n,i\leq 2}(1+n^4)|\int dx e^{ikx}\int dy(1-\partial_x^2)(\chi_n(x)\partial_x^i G(x,y))g(y)|. \tag{55}$$

We consider first $|n| \geq C\log L$. Then

$$\begin{aligned}|\ldots| &\leq \sum_m \int dxdy|(1-\partial_x^2)(\chi_n(x)\partial_x^i G(x,y))\chi_m(y)g(y)| \\ &\leq C\sum_m e^{-|n-L^{-1}m|}(1+m^4)^{-1}\|g\| \leq CL^{-3}\frac{\|g\|}{(1+n^4)}\end{aligned} \tag{56}$$

for L large enough. We used (41) (with $\partial^i w$ replaced by g) and (54). To control the sum, we used $|n| \geq C\log L$ for $|n - L^{-1}m| \geq \frac{|n|}{2}$, and $(1+m^4)^{-1}$ for $|n-L^{-1}m| \leq \frac{|n|}{2}$, which implies $m \geq L\frac{|n|}{2}$.

For $|n| \leq C\log L$, we subtract $G(x,0)$ from $G(x,y)$ in (55). We may do this for free, since $\int g(y)dy = 0$. Then,

$$|\ldots| \leq \sum_m \int dxdy|(1-\partial_x^2)(\chi_n(x)\partial_x^i(G(x,y)-G(x,0)))\chi_m(y)g(y)|. \tag{57}$$

For the terms with $|m| \leq L$, we use

$$|\partial^j(G(x,y)-G(x,0))| \leq CL^{-1}e^{-|n|}(1+|m|) \tag{58}$$

and thus, using (41) again,

$$\sum_{|m|\leq L} \leq CL^{-1} \sum_{|m|\leq L} e^{-|n|} \frac{\|g\|}{1+|m|^3} \leq CL^{-1} \frac{\|g\|}{1+n^4}. \tag{59}$$

For the terms with $|m| \geq L$, the subtraction in (57) is not necessary, (41) suffices.

$$\sum_{|m|\geq L} \leq C \sum_{|m|\geq L} \frac{\|g\|}{1+m^4} \leq CL^{-3}\|g\| \leq CL^{-1} \frac{\|g\|}{1+n^4}. \tag{60}$$

for L large, since $|n| \leq C \log L$. Hence, (52) follows.

(c) *Iteration.* To conclude the proof, we need to iterate R. Going back to (31), we study R_{L,K_n}. From the expression (30) for K and from (32), we see that the only change to the $n=1$ analysis above will be a change in (51):

$$\|R_{L,K_n} g_n\| \leq \frac{C}{L}\|g_n\| + C_{L,a}\epsilon(L^{-n}\epsilon + \|g_n\|). \tag{61}$$

Indeed, all the terms in (30) are irrelevant, except some of those with $l \geq 1, N = 1, M = 0, 1$, which are bounded by $C_{L,a}\epsilon\|g_n\|$. Hence, we get (recall that $\|g\| \leq C\|f\|$ by (28))

$$\|R_{L^n,K} g\| \leq C_{L,a} L^{-(1-\delta)n}(\|f\| + \epsilon) \tag{62}$$

which allows us to conclude the proof as before. □

Proof of the Proposition. To solve (10), we set

$$h = A\psi \tag{63}$$

and solve

$$h = \partial(a^*(\phi_0^* + A^{-1}h)\partial(\phi_0^* + A^{-1}h)) \equiv \psi_0 + N(h) \tag{64}$$

in the Banach space (13). We need some information on the operator A. In fact, A is just the harmonic oscillator in disguise:

$$e^{\frac{x^2}{8}} A e^{-\frac{x^2}{8}} = -\partial^2 + \frac{x^2}{16} + \frac{1}{4}. \tag{65}$$

Hence, the kernel of A^{-1} is readily computed from Mehler's formula [19]. The result is

$$A^{-1}(x, y) \equiv M(x, y) = \int_0^\infty dt M_t(x, y) \tag{66}$$

with

$$M_t(x, y) = \frac{e^{-\frac{t}{2}}}{\pi^{\frac{1}{2}}(1-e^{-t})^{\frac{1}{2}}} e^{-\frac{1}{4}\frac{(x-e^{-\frac{t}{2}}y)^2}{1-e^{-t}}}. \tag{67}$$

In $N(h)$ we encounter terms $\partial A^{-1}h$ and $\partial^2 A^{-1}h$. We write the latter as $\partial^2 A^{-1}h = (-A - \frac{1}{2}x\partial)A^{-1}h = -h - \frac{1}{2}x\partial A^{-1}h$. Thus,

$$N(h) = R(A^{-1}h, \partial A^{-1}h, x\partial A^{-1}h) \tag{68}$$

with $R : \mathbf{C}^3 \to \mathbf{C}$ analytic near zero. Thus all we now need is

Lemma. *The operators A^{-1}, ∂A^{-1} and $x\partial A^{-1}$ are continous in the norm* (13).

Proof. We want to show

$$|(1+|x|)\partial^i(A^{-1}h)(x)| \leq Ce^{-\frac{x^2}{8}}\|h\|_N \ , \ i \leq N+1. \tag{69}$$

Let

$$M = \int_0^\tau M_t dt + \int_\tau^\infty M_t dt \equiv M^{(1)} + M^{(2)} \tag{70}$$

For $M^{(2)}$ we have

$$|\partial^i(M^{(2)}h)(x)| \leq \int_\tau^\infty dt \int dy |\partial^i M_t(x,y)| e^{-\frac{y^2}{8}}\|h\|_N \tag{71}$$

From (67) we have the estimate, for any $\delta > 0$,

$$|\partial^i M_t(x,y)| \leq C\delta^{-\frac{1}{2}i} e^{-\frac{t}{2}} (1-e^{-t})^{-\frac{1}{2}(i+1)} e^{-\frac{1-\delta}{4}\frac{(x-e^{-\frac{t}{2}}y)^2}{1-e^{-t}}} \tag{72}$$

Thus, using

$$\int e^{-\frac{1-\delta}{4}\frac{(x-e^{-\frac{t}{2}}y)^2}{1-e^{-t}} - \frac{y^2}{8}} dy = 2\sqrt{\pi}((1-\delta)\frac{e^{-t}}{1-e^{-t}} + \frac{1}{2})^{-\frac{1}{2}} e^{-\frac{1-\delta}{4}\frac{x^2}{1+(1-2\delta)e^{-t}}} \tag{73}$$

we see that for τ large enough and δ small,

$$|(1+|x|)\partial^i(M^{(2)}h)(x)| \leq C\|h\|_N e^{-\frac{x^2}{8}} \tag{74}$$

For $M^{(1)}$ we have (let $i = N+1$, the worst case), integrating by parts,

$$|\partial^{N+1}(M^{(1)}h)(x)| = |\int_0^\tau dt e^{N\frac{t}{2}} \int dy \partial_x M_t(x,y)\partial^N h| \tag{75}$$

and thus by (72) and (73) this is bounded by

$$|\partial^{N+1}(M^{(1)}h)(x)| \leq \frac{C}{\sqrt{\delta}}\|h\|_N \int_0^\tau dt \frac{e^{-\frac{t}{2}}}{(1-e^{-t})^{1/2}} e^{-\frac{1-\delta}{4}\frac{x^2}{1+(1-2\delta)e^{-t}}} f(t) \tag{76}$$

where $f(t) = (\frac{1}{2}(1-e^{-t}) + (1-\delta)e^{-t})^{-1/2}$ is $\mathcal{O}(1)$ for all t. Now, using $e^{-t} \leq (1+t)^{-1}$, we have

$$-\frac{1-\delta}{4}\frac{x^2}{1+(1-2\delta)e^{-t}} \leq -\frac{x^2}{8} - t\frac{x^2}{8}(\frac{1-2\delta}{2-2\delta+t}) \tag{77}$$

and so we get from the t integral

$$\int_0^\tau dt \frac{e^{-ctx^2-\frac{t}{2}}}{(1-e^{-t})^{1/2}} \leq C|x|^{-1} \tag{78}$$

yielding

$$|(1+|x|)\partial^{N+1}(M^{(1)}h)(x)| \leq C\|h\|_N e^{-\frac{x^2}{8}}.$$

For $j < N+1$, write

$$|\partial^j(M^{(1)}h)(x)| = |\int_0^\tau dt e^{j\frac{t}{2}} \int dy M_t(x,y)\partial^j h| \tag{79}$$

and get, from (72) for $i=0$ and from (78),

$$|(1+|x|)\partial^j(M^{(1)}h)(x)| \leq C\|h\|_N e^{-\frac{x^2}{8}}. \tag{80}$$

The Lemma is proved.

Solving (64) is now trivial. We have

$$\|\psi_0\|_N \leq C_a \epsilon^{d+1} \tag{81}$$

and

$$\|N(h)\|_N \leq C_a(\epsilon^d\|h\|_N + \|h\|_N^2)$$

By the Lemma then

$$\|\psi\|_N \leq C\|h\|_N$$

and the Proposition is proved. □

The perturbed Burgers' equation

Let us now discuss the perturbed Burgers' equation, (3.36). To understand the relevant fixed point, consider first the case $H=0$. Then, by the Cole-Hopf transformation [6], the Burgers' equation reduces to the heat equation : set

$$\psi(x,t) = e^{\int_{-\infty}^x u(y,t)dy} \tag{82}$$

Then, $\dot\psi = \psi''$ if

$$\dot u = u'' + (u^2)' \tag{83}$$

Now, $\lim_{x\to-\infty}\psi(x,1) = 1$, $\lim_{x\to+\infty}\psi(x,1) = e^{\int_{-\infty}^{+\infty} u(y,1)dy}$, so that we look for a fixed point for ψ of the form (3). Using the fact that the inverse transformation of (82) is $u(x,t) = \frac{\psi'(x,t)}{\psi(x,t)}$, we get the family of fixed points

$$f_A^*(x) = \frac{Ae'(x)}{1+Ae(x)} = \frac{d}{dx} \log(1+Ae(x)) \tag{84}$$

for the transformation (2.4) with $\alpha = 1$, where u solves (83).

Note that the derivative in (84) explains why f_A^* is a fixed point of (2.4) with $\alpha = 1$ while ϕ_0^* in (3) is a fixed point of (4) (where $\alpha = 0$).

Since $e(-\infty) = 0, e(+\infty) = 1$,

$$\log(1 + A) = \int_{-\infty}^{+\infty} f_A^*(x)\, dx \tag{85}$$

Now we can state

Theorem 4 *Let $H : \mathbf{C}^3 \to \mathbf{C}$ be analytic in a neighbourghood of* 0 *with $d_F > 0$ and fix a $\delta > 0$. Then, there exists an $\epsilon > 0$ such that, if $\|f\| \leq \epsilon$, equation* (3.36) *with $u(x, 1) = f(x)$ has a unique solution which satisfies*

$$\lim_{t\to\infty} t^{1-\delta} \|u(\cdot t^{1/2}, t) - t^{-1/2} f_A^*(\cdot)\| = 0$$

(the norms here are defined by (3.3)*).*

Proof. The proof is very similar to the one of Theorem 1, except for the change of fixed point. Let

$$f(x) = f_{A_0}^*(x) + g_0(x) \tag{86}$$

with

$$\log(1 + A_0) = \int_{-\infty}^{+\infty} f(x)\, dx \tag{87}$$

so that, by (85),

$$\int_{-\infty}^{+\infty} g_0(x)\, dx = \hat{g}_0(0) = 0 \tag{88}$$

We have $\|g_0\| \leq C\|f\|$: using $e'(x) = \frac{e^{-x^2/4}}{\sqrt{4\pi}}$, we see that $f_{A_0}^*$ and all its derivatives are integrable; using $\|\hat{h}\|_\infty \leq \|h\|_1$ we have

$$\|f_{A_0}^*\| \leq C|A_0| \tag{89}$$

Also,

$$|A_0| \leq C\|f\| \tag{90}$$

for $\|f\| \leq \epsilon$, because, by (87), $|A_0| \leq C\|f\|_1 \leq C\|f\|$. For the last inequality, we use $\|f\|_1 \leq C\|(1 + |x|)f\|_2$, by Schwartz' inequality, and $\|f\|_2 + \|xf\|_2 \leq C\|f\|$, by Plancherel.

The local existence is proven as before. To study R_L, let us write the solution $u(x, t)$ as

$$u(x, t) = u_{A_0}^*(x, t) + v(x, t)$$

with

$$u^*_{A_0}(x,t) = t^{-1/2} f^*_{A_0}(\frac{x}{\sqrt{t}})$$

Since $u^*_{A_0}(x,t)$ solves (83) with $u^*_{A_0}(x,1) = f^*_{A_0}(x)$, $v(x,t)$ satisfies :

$$\dot{v} = v'' + ((u^*_{A_0} + v)^2 - (u^*_{A_0})^2)' + H(u,u',u'') \tag{91}$$

with

$$v(x,1) = g_0(x) \tag{92}$$

Now,

$$R_L\ f(\cdot) = Lu(L\cdot, L^2) = f^*_{A_0}(\cdot) + L\ v(L\cdot, L^2).$$

Write it as

$$R_L\ f(\cdot) = f^*_{A_1}(\cdot) + g_1(\cdot) \tag{93}$$

with $\int_{-\infty}^{+\infty} g_1(x)\ dx = 0$.

This means that we define A_1 (see (85)) by

$$\log(\frac{1+A_1}{1+A_0}) = \int_{-\infty}^{+\infty} v(x, L^2)\ dx \tag{94}$$

But, by (88), (92), $\int v(x,1)\ dx = 0$ and, by (91), $|\frac{d}{dt}\int v(x,t)dx| \leq \int |H(u,u',u'')|dx \leq \|H(u,u',u'')\|$ which, by a bound similar to (3.14),(3.17), gives

$$|\int_{-\infty}^{+\infty} v(x,L^2)dx| \leq C_{L,H}\|f\|^2 \tag{95}$$

So, inserting (95) in (94), we have, for $\|f\|$ small,

$$|A_1 - A_0| \leq C_{L,H}\|f\|^2 \tag{96}$$

We also have

$$\|g_1\| \leq CL^{-1}\|g_0\| + C_{L,H}\|f\|^2 \leq L^{-(1-\delta)}\|f\| \tag{97}$$

This is like (3.25) : writing (91) as an integral equation we have $Lv(L\cdot, L^2) = R_0 g_0 +$ rest; $R_0 g_0$ contracts because of (3.4) and the rest is the sum of two terms which are bounded using (3.13): the first, coming from $2(u^*_{A_0} v)'$, is less than $C\epsilon\|g_0\| \leq L^{-1}\|g_0\|$, (because, using (90), $\|u^*_{A_0}\|_L + \|(u^*_{A_0})'\|_L \leq C\|f\| \leq C\epsilon$), and the second (coming from $(v^2)' + H$) is bounded by $C_{L,H}\|f\|^2$.

Now the iteration is exactly as before. The H term runs down with L^{-nd_H} so that, using (95), we have

$$|A_{n+1} - A_n| \leq C_{L,H} L^{-nd_H}\|f\|^2$$

and, as in (97),

$$\|g_n\| \leq CL^{-(1-\delta)n}\|f\|.$$

The rest of the proof is as in Theorem 1. □

5. Discussion

Now, we want to put our results in a more general framework, relate them to other works, and explain the analogy with the theory of critical phenomena. The discussion will be heuristic or based on previously known results, and we shall limit ourselves to the family of heat equations with absorption:

$$\dot{u} = u'' - u^p \tag{1}$$

where $p > 1$ is not necessarily an integer. In our terminology, $p > 3$ is irrelevant, $p = 3$ marginal, and $p < 3$ relevant; 3 has to be replaced by $1 + \frac{2}{N}$ in N dimensions.

To understand the $p < 3$ case, observe that (1) is invariant under the scaling transformation (2.3) for $\alpha = \frac{2}{p-1}$. Finding a fixed point for that transformation amounts to finding a solution of the form $u(x,t) = t^{-\frac{1}{p-1}} f^*(\frac{x}{\sqrt{t}})$, where f^* is a solution of the ODE (replacing $\frac{x}{\sqrt{t}}$ by x) :

$$f'' + \frac{1}{2}xf' + \frac{f}{p-1} - f^p = 0 \tag{2}$$

The following results are known about the solutions of (2):

1) For any $1 < p < 3$, there exists an everywhere positive solution f_1^* of (2), which has almost Gaussian decay at infinity [3, 11].

2) For any $p > 1$, there exists a solution f_2^* which decays at infinity like $|x|^{-\frac{2}{p-1}}$ [11, 16]. Note that f_2^* is integrable only for $p < 3$.

Rather detailed results are known on the basin of attraction of the various fixed points; we state them loosely, see the references for more precise statements e.g. on the type of convergence; also, in each case, the decay in time of $u(x,t)$ is $t^{\frac{-\alpha}{2}}$, as in (2.10), where the exponent α is related to the fixed point as in (2.3, 2.4).

1) For $p \geq 3$, and initial data $u(x,1)$ non-negative and integrable, the asymptotic behaviour of the solution is governed by f_0^*, as in Theorems 1, 2, with logarithmic corrections for $p = 3$ [11, 12]. This is a global result ($u(x,1)$ is not assumed to be small); our results are perturbative, and restricted to integer p (but a general non-linearity F), but do not require $u(x,1)$ to be pointwise positive and hold (when $p > 3$) also if one has $+u^p$ in (1); in that case, smallness of $u(x,1)$ is necessary since large initial data blow up in a finite time [10, 9, 18].

2) For $p < 3$ and $u(x,1)$ non-negative and having (suitable) Gaussian decay, the asymptotic behaviour is governed by the non-trivial fixed point f_1^* [11, 17].

3) If one starts with $u(x,1)$ non-integrable and decaying at infinity like $|x|^{-\alpha}$, with $0 < \alpha < 1$, then, for $p > 3$, the relevant "Gaussian" fixed point is f_α^* where $\hat{f}_\alpha^*(k) = |k|^{\alpha-1}e^{-k^2}$, which, for any α, is a fixed point of (2.4), where u solves the heat equation. This fixed point has the right $|x|^{-\alpha}$ decay at infinity. Now u^p is relevant, marginal, or irrelevant according to whether $p < 1+\frac{2}{\alpha}$, $p = 1+\frac{2}{\alpha}$, or $p > 1+\frac{2}{\alpha}$. One knows [16, 12] that, for a non-negative initial data, the solution converges to f_α^* for $p > 1+\frac{2}{\alpha}$. For $p = 1+\frac{2}{\alpha}$, it converges to f_2^*. For $p < 1+\frac{2}{\alpha}$, the solution converges to a solution constant in space, $(p-1)^{-\frac{1}{p-1}}t^{-\frac{1}{p-1}}$, which solves (1) without the diffusive term: $\dot{u} = -u^p$, and which can be viewed as a (somewhat degenerate) new fixed point.

To see the analogy with the theory of critical phenomena, consider an Ising model or a ϕ^4 theory, on an N dimensional lattice, at the critical point. $N > 4$ is like $p > 3$ here: ϕ^4 is irrelevant and the behaviour at the critical point is governed by the Gaussian fixed point. For $N = 4$, ϕ^4 becomes marginal, and the Gaussian behaviour is modified, like here in Theorem 2, by logarithmic corrections. However, this is not true for every marginal perturbation. In ϕ^4 theory, like here for $p = 3$, this happens because the marginal term becomes irrelevant when higher order terms are included: A_n and, therefore, f_n go to zero which is the same thing as having a coupling constant in front of the u^3 term going to zero. In particular, this higher order irrelevancy depends, like in ϕ^4 theory, on the sign of the perturbation. For $+u^3$ in (1), the solution blows up [9, 18]. In point 3) above, the marginal perturbation ($p = 1+\frac{2}{\alpha}$), leads to a non-trivial fixed point, f_2^* instead of f_α^*. Also, in the Burgers' equation (3.36), the marginal term remains marginal to all orders and the solution is governed by a new fixed point, which is however easy to write down (see the end of Sect.4).

For $N < 4$, ϕ^4 becomes relevant and one expects the critical behaviour to be governed by a non-trivial fixed point, whose existence is however much harder to establish than here. Another analogy with field theory concerns the constant A in (7): this is like a "renormalised" constant whose corresponding "bare" value is $A_0 = \hat{f}(0)$. One of the problems encountered in proving (7), instead of just a bound on $u(x,t)$, is that A, unlike A_0, may depend in a complicated way on f and F, and is not known explicitely. This usually limits the power of ordinary perturbation theory: one may try to expand $u(x,t)$ around $A_0 t^{-\frac{1}{2}} f_0^*$ which has the wrong constant and the perturbation series may not converge. One of the advantages of the RG method is that it allows to "build up" A through a convergent sequence of approximations, as we did in the proof. Also, note that in the marginal case, the renormalisation of A drives it to zero and produces the logarithmic correction in (39).

Finally, one may also interpret in RG language the results on the existence of singular or very singular solutions of (1). A solution is *singular* if, as $t \to 0$, $u(x,t)$ becomes concentrated on a point, and it is *very singular* if, moreover, $\int dx u(x,t)$ diverges when $t \to 0$. For the family of equations (1), one knows that, for $1 < p < 3$, there exists both a singular and a very singular solution [3, 4]; the singular solution is the fundamental one while $u(x,t) = t^{-\frac{1}{p-1}} f_1^*(\frac{x}{\sqrt{t}})$ is very singular, since f_1^* is integrable. Besides, the fundamental solution behaves, for $t \to 0$, like the corresponding solution of the heat equation $(t^{-\frac{1}{2}} f_0^*(\frac{x}{\sqrt{t}}))$. On the other hand, for $p \geq 3$, there does not exist any singular

solution [4].

This can be understood in RG language: start with some non-singular data at $t = 1$; a singular solution would be obtained by solving (1) backwards in time and letting $t \to 0$. This amounts to running backwards the RG flow; hence, the stability of the fixed points is inverted: f_0^* becomes stable for $1 < p < 3$ and unstable for $p \geq 3$. So, for $p < 3$, one would expect the solution to be attracted to f_0^* when $t \to 0$, thus explaining the presence of a fundamental solution. On the other hand, for $p \geq 3$, one could only go, when $t \to 0$, towards f_2^*, which, because of its power law decay at infinity, cannot be a singular solution (it is not integrable for any t). This correspondence between $t \to 0$ and $t \to \infty$ is similar to the correspondence between "ultraviolet" and "infrared" behaviour of fixed points in field theory and in critical phenomena.

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